Math 102

Krishanu Sankar

September 13, 2018

Announcements

- WeBWork Diagnostic Test (due Sunday) - you get 100% of points just for completing it.

Goals Today

- Average rates of change
  - Calculating them from data
  - Graphically - secant lines
- Using a spreadsheet for computation
- Instantaneous rates of change (derivatives)
  - Computed from average rates of change using a limit
  - Computing the derivative
  - Graphically - tangent lines

Average Rate of Change

- Let $P(t)$ be a function of time, t. The **average rate of change** of $P(t)$ from $t = a$ to $t = b$ is defined to be

$$\frac{\text{Change in } P}{\text{Change in } t} = \frac{\Delta P}{\Delta t} = \frac{P(b) - P(a)}{b - a}$$

Average Rate of Change

- Let $P(t)$ be a function of time, t. The **average rate of change** of $P(t)$ from $t = a$ to $t = b$ is defined to be

$$\frac{\text{Change in } P}{\text{Change in } t} = \frac{\Delta P}{\Delta t} = \frac{P(b) - P(a)}{b - a}$$

- Example: if $P(t)$ is the **position of an object**, then the above formula gives the **average velocity** over a period of time.

Example - Average Velocity

Distance	20m	30m	40m	60m	80m	100m
Time	2.89	3.79	4.64	6.31	7.92	9.58

Jamaican sprinter Usain Bolt's split times in his world record run at the 2009 World Championships Men's 100m in Berlin.

Source: https://biomech.byu.edu/Portals/82/docs/coaching/100m%20WR%20Split%20Analysis.xlsx

Picture source: https://sg.news.yahoo.com/world-record-bolt-triumphed-over-air-231023749.html

Distance	20m	30m	40m	60m	80m	100m
Time	2.89	3.79	4.64	6.31	7.92	9.58

- Question: How would we calculate Bolt's average velocity from 20m to 40m? 40m to 60m? 20m to 60m? 50m to 70m?

Distance	20m	30m	40m	60m	80m	100m
Time	2.89	3.79	4.64	6.31	7.92	9.58

- Question: How would we calculate Bolt's average velocity from 20m to 40m? 40m to 60m? 20m to 60m? 50m to 70m?

$$\text{Average velocity} = \frac{\text{Distance traveled}}{\text{Time elapsed}}$$

Distance	20m	30m	40m	60m	80m	100m
Time	2.89	3.79	4.64	6.31	7.92	9.58

- Question: How would we calculate Bolt's average velocity from 20m to 40m? 40m to 60m? 20m to 60m? 50m to 70m?

$$\text{Average velocity} = \frac{40-20}{4.64-2.89}\ \mathrm{m/s}$$

Warning: remember which term goes in the numerator and which goes in the denominator! This is the average velocity from $t = 2.89$ to $t = 4.64$!

Distance	20m	30m	40m	60m	80m	100m
Time	2.89	3.79	4.64	6.31	7.92	9.58

- Question: How would we calculate Bolt's average velocity from 20m to 40m? 40m to 60m? 20m to 60m? 50m to 70m?

$$\text{Average velocity} = \frac{60-40}{6.31-4.64}\ \mathrm{m/s}$$

Warning: remember which term goes in the numerator and which goes in the denominator! This is the average velocity from $t = 2.89$ to $t = 6.31$!

Distance	20m	30m	40m	60m	80m	100m
Time	2.89	3.79	4.64	6.31	7.92	9.58

- Question: How would we calculate Bolt's average velocity from 20m to 40m? 40m to 60m? 20m to 60m? 50m to 70m?

$$\text{Average velocity} = \frac{60-20}{6.31-2.89}\ \mathrm{m/s}$$

Warning: remember which term goes in the numerator and which goes in the denominator! This is the average velocity from $t = 4.64$ to $t = 6.31$!

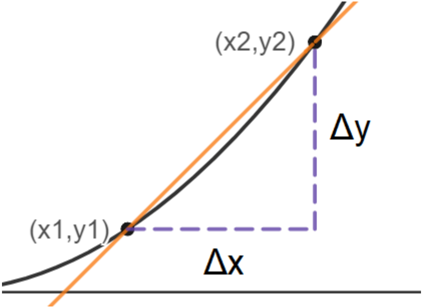
Distance	20m	30m	40m	60m	80m	100m
Time	2.89	3.79	4.64	6.31	7.92	9.58

- Question: How would we calculate Bolt's average velocity from 20m to 40m? 40m to 60m? 20m to 60m? 50m to 70m?

Not enough information to calculate.

Graphically - Secant Lines

The secant line to a curve through two points P and Q is the line passing through those two points.



https://www.desmos.com/calculator/0gbqdldt1v

Its slope is rise over run.

$$\text{Slope} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

Instantaneous rate of change

Suppose that the height in meters of a rising balloon is given by the equation $P(t) = t^2$.

Instantaneous rate of change

Suppose that the height in meters of a rising balloon is given by the equation $P(t) = t^2$.

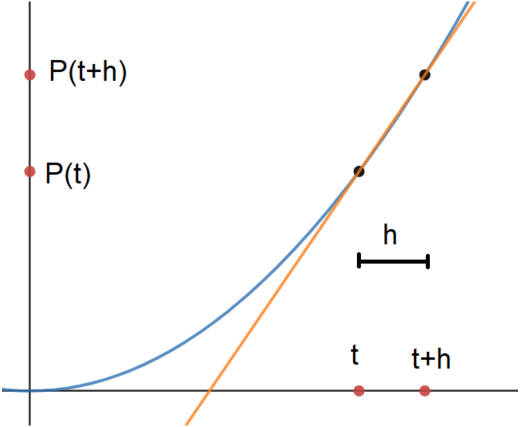
- Exercise: What is the average velocity of the balloon from $t = 5$ to $t = 7$?
- Exercise: What is the average velocity of the balloon from $t = 5$ to $t = 6$?

Instantaneous rate of change

Suppose that the height in meters of a rising balloon is given by the equation $P(t) = t^2$.

- Exercise: What is the average velocity of the balloon from $t = 5$ to $t = 7$?
- Exercise: What is the average velocity of the balloon from $t = 5$ to $t = 6$?
- $t = 5$ to $t = 5.1$? $t = 5$ to $t = 5.01$?
  https://docs.google.com/spreadsheets/d/18GnyHRe2T4AxrEgqXmo0LZsYB-0pF_it08vlR6sJtME/edit?usp=sharing

Instantaneous Rate of Change



We consider the interval from $t = 5$ to $t = 5 + h$, and approximate the average rate of change **when** h **is very small.** (asymptotic thinking!)

Instantaneous Rate of Change

We consider the interval from $t = 5$ to $t = 5 + h$, and approximate the average rate of change **when** h **is very small.** (asymptotic thinking!)

$$\frac{\Delta P}{\Delta t} = \frac{P(5+h) - P(5)}{(5+h) - 5} = \frac{(5+h)^2 - 5^2}{h}$$

Instantaneous Rate of Change

We consider the interval from $t = 5$ to $t = 5 + h$, and approximate the average rate of change **when** h **is very small.** (asymptotic thinking!)

$$\frac{\Delta P}{\Delta t} = \frac{P(5+h) - P(5)}{(5+h) - 5} = \frac{(5+h)^2 - 5^2}{h}$$

$$= \frac{(25 + 10h + h^2) - 25}{h} = \frac{10h + h^2}{h} = 10 + h$$

Instantaneous Rate of Change

We consider the interval from $t = 5$ to $t = 5 + h$, and approximate the average rate of change **when** h **is very small.** (asymptotic thinking!)

$$\frac{\Delta P}{\Delta t} = \frac{P(5+h) - P(5)}{(5+h) - 5} = \frac{(5+h)^2 - 5^2}{h}$$

$$= \frac{(25 + 10h + h^2) - 25}{h} = \frac{10h + h^2}{h} = 10 + h$$

When h is **very small**, this is approximately 10.

Instantaneous Rate of Change

We consider the interval from $t = 5$ to $t = 5 + h$, and approximate the average rate of change **when** h **is very small.** (asymptotic thinking!)

$$\frac{\Delta P}{\Delta t} = \frac{P(5+h) - P(5)}{(5+h) - 5} = \frac{(5+h)^2 - 5^2}{h}$$

$$= \frac{(25 + 10h + h^2) - 25}{h} = \frac{10h + h^2}{h} = 10 + h$$

When h is **very small**, this is approximately 10.

$$\lim_{h \to 0} (10 + h) = 10$$

'The limit as h goes to 0 of $10 + h$ equals 10.'

Instantaneous Rate of Change

- Let $f(x)$ be a function, and let x_0 be some number. The **instantaneous rate of change** of $f(x)$ is defined as the **limit** (if it exists)

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

What is a limit anyways? To be addressed in upcoming classes!

Instantaneous Rate of Change

- Let $f(x)$ be a function, and let x_0 be some number. The **instantaneous rate of change** of $f(x)$ is defined as the **limit** (if it exists)

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

What is a limit anyways? To be addressed in upcoming classes!

- **Warning:** $\lim\limits_{h \to 0}$ is NOT the same as plugging in $h = 0$!

Instantaneous Rate of Change

- Let $f(x)$ be a function, and let x_0 be some number. The **instantaneous rate of change** of $f(x)$ is defined as the **limit** (if it exists)

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

What is a limit anyways? To be addressed in upcoming classes!

- **Warning:** $\lim\limits_{h \to 0}$ is NOT the same as plugging in $h = 0$!
- Graphically, this is represented as the **slope of the tangent line at** $(x_0, f(x_0))$**.**

Now you try!

- Exercise: Calculate the instantaneous rate of change of $f(x) = x^2$ at $x = 3$. **Do this using the definition as a limit.**
- Further Exercise: Try for $f(x) = x^3$.
- Question: If we let h be negative, how does the picture change?

The Derivative

Let $f(x)$ be a function. The **derivative of** f is a new function, written as f', whose value at a point x_0 is given by

$$f'(x_0) = \text{Instantaneous rate of change of } f \text{ at } x_0$$

$$= \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

Terminology

Question: The **derivative** of f, evaluated at $x = x_0$, equals

(A) $\frac{f(x_0+h)-f(x_0)}{h}$

(B) The slope of the tangent line at $x = x_0$

(C) $\lim\limits_{x_0 \to 0} \frac{f(x_0+h)-f(x_0)}{h}$

(D) The line intersecting the graph of the function at $x = x_0$ and $x = x_0 + h$

(E) More than one of the above

(F) None of the above

Terminology

Question: The **derivative** of f, evaluated at $x = x_0$, equals

(A) $\frac{f(x_0+h)-f(x_0)}{h}$

(B) The slope of the tangent line at $x = x_0$

(C) $\lim\limits_{x_0 \to 0} \frac{f(x_0+h)-f(x_0)}{h}$

(D) The line intersecting the graph of the function at $x = x_0$ and $x = x_0 + h$

(E) More than one of the above

(F) None of the above

Terminology

Question: Which of the following is a **secant line**?

(A) $\frac{f(x_0+h)-f(x_0)}{h}$

(B) The slope of the tangent line at $x = x_0$

(C) $\lim\limits_{x_0 \to 0} \frac{f(x_0+h)-f(x_0)}{h}$

(D) The line intersecting the graph of the function at $x = x_0$ and $x = x_0 + h$

(E) More than one of the above

(F) None of the above

Terminology

Question: Which of the following is a **secant line**?

(A) $\frac{f(x_0+h)-f(x_0)}{h}$

(B) The slope of the tangent line at $x = x_0$

(C) $\lim\limits_{x_0 \to 0} \frac{f(x_0+h)-f(x_0)}{h}$

(D) The line intersecting the graph of the function at $x = x_0$ and $x = x_0 + h$

(E) More than one of the above

(F) None of the above

Terminology

Question: The **total change** of $f(x)$ over the interval $[x_0, x_0 + h]$ equals

(A) $\frac{f(x_0+h)-f(x_0)}{h}$

(B) The slope of the tangent line at $x = x_0$

(C) $\lim\limits_{x_0 \to 0} \frac{f(x_0+h)-f(x_0)}{h}$

(D) The line intersecting the graph of the function at $x = x_0$ and $x = x_0 + h$

(E) More than one of the above

(F) None of the above

Terminology

Question: The **total change** of $f(x)$ over the interval $[x_0, x_0 + h]$ equals

(A) $\frac{f(x_0+h)-f(x_0)}{h}$

(B) The slope of the tangent line at $x = x_0$

(C) $\lim\limits_{x_0 \to 0} \frac{f(x_0+h)-f(x_0)}{h}$

(D) The line intersecting the graph of the function at $x = x_0$ and $x = x_0 + h$

(E) More than one of the above

(F) None of the above

Terminology

Question: The **average rate of change** of $f(x)$ over the interval $[x_0, x_0 + h]$ equals

(A) $\frac{f(x_0+h)-f(x_0)}{h}$

(B) The slope of the tangent line at $x = x_0$

(C) $\lim\limits_{x_0 \to 0} \frac{f(x_0+h)-f(x_0)}{h}$

(D) The line intersecting the graph of the function at $x = x_0$ and $x = x_0 + h$

(E) More than one of the above

(F) None of the above

Terminology

Question: The **average rate of change** of $f(x)$ over the interval $[x_0, x_0 + h]$ equals

(A) $\frac{f(x_0+h)-f(x_0)}{h}$

(B) The slope of the tangent line at $x = x_0$

(C) $\lim_{x_0 \to 0} \frac{f(x_0+h)-f(x_0)}{h}$

(D) The line intersecting the graph of the function at $x = x_0$ and $x = x_0 + h$

(E) More than one of the above

(F) None of the above

Recap and Reminders

- Average rates of change
  - Calculating them from data
  - Graphically - secant lines
- Using a spreadsheet for computation
- Instantaneous rates of change (derivatives)
  - Computed from average rates of change using a limit
  - Computing the derivative
  - Graphically - tangent lines
- WeBWork Diagnostic Test due Sunday

Bonus slide - Derivative of $f(x) = x^2$ evaluated at $x = x_0$

$$
\begin{aligned}
f'(x_0) &= \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} \\
&= \lim_{h \to 0} \frac{(x_0 + h)^2 - x_0^2}{h} \\
&= \lim_{h \to 0} \frac{(x_0^2 + 2x_0 h + h^2) - x_0^2}{h} \\
&= \lim_{h \to 0} \frac{2x_0 h + h^2}{h} \\
&= \lim_{h \to 0} (2x_0 + h) = 2x_0
\end{aligned}
$$

So $f'(3) = 2 \cdot 3 = 6$.

Bonus slide - Derivative of $f(x) = x^3$ evaluated at $x = x_0$

$$
\begin{aligned}
f'(x_0) &= \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} \\
&= \lim_{h \to 0} \frac{(x_0 + h)^3 - x_0^3}{h} \\
&= \lim_{h \to 0} \frac{(x_0^3 + 3x_0^2 h + 3x_0 h^2 + h^3) - x_0^3}{h} \\
&= \lim_{h \to 0} \frac{3x_0^2 h + 3x_0 h^2 + h^3}{h} \\
&= \lim_{h \to 0} (3x_0^2 + 3x_0 h + h^2) = 3x_0
\end{aligned}
$$